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A HOMOTOPY CLASSIFICATION OF CERTAIN 7-MANIFOLDS

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Abstract. This paper gives a homotopy classification of Wallach spaces and a partial homotopy classification of closely related spaces obtained by free S^1-actions on $SU(3)$ and on $S^3 \times S^5$.

Introduction

In the class of 1-connected, compact homogeneous spaces Kreck and Stolz [KS1], [KS2] were the first to discover examples of homeomorphic but not diffeomorphic spaces. Some of these examples occur in the infinite family of Wallach spaces. For coprime integers n, m let

$$\Delta_{m,n} : S^1 \to SU(3)$$

be the homomorphism $z \mapsto diag(z^n, z^m, z^{-m-n})$. Then Wallach spaces are the 7-dimensional homogeneous manifolds $SU(3)/\Delta_{m,n}(S^1)$. Kreck and Stolz introduced invariants which determine the homeomorphism and diffeomorphism type of Wallach spaces. The main motivation of this paper is the homotopy classification of Wallach spaces. The following theorem is proved.

Theorem 0.1. *$SU(3)/\Delta_{m,n}(S^1)$ and $SU(3)/\Delta_{\tilde{m},\tilde{n}}(S^1)$ are homotopy equivalent if and only if the following conditions hold:*
i) $r := n^2 + nm + m^2 = \tilde{n}^2 + \tilde{m}\tilde{n} + \tilde{m}^2$
ii) $nm(n+m) \equiv \pm\tilde{n}\tilde{m}(\tilde{n}+\tilde{m})$ *mod* r.

These conditions look quite similar to those of Kreck and Stolz's [KS2]: Letting i) unaltered and replacing ii) by $nm(n+m) \equiv \pm\tilde{n}\tilde{m}(\tilde{n}+\tilde{m})$ mod $24r$, one obtains the necessary and sufficient conditions for homeomorphy.

In this paper we also study the homotopy type of a more general class of 1-connected compact manifolds having the same cohomology ring structure as Wallach spaces, i.e. $H^0 \cong \mathbb{Z}$, $H^1 = 0$, $H^2 \cong \mathbb{Z}$ with a generator u, $H^3 = 0$, $H^4 \cong \mathbb{Z}_r$ is a cyclic group of order r generated by u^2, $H^5 \cong \mathbb{Z}$ with a generator x_5, $H^6 = 0$, $H^7 \cong \mathbb{Z}$ is generated by $x_5 \cdot u$. These manifolds will be called manifolds of type r. For example, consider the family of S^1-actions $S^1 \times SU(3) \to SU(3)$,

$$(z, A) \mapsto diag(z^{k_1}, z^{k_2}, z^{k_3})A\ diag(z^{-l_1}, z^{-l_2}, z^{-l_3}),$$

parameterized by $k := (k_1, k_2, k_3)$, $l := (l_1, l_2, l_3) \in \mathbb{Z}^3$, whose orbit spaces $N_{k,l}$ are manifolds of type $r := k_1k_2+k_1k_3+k_2k_3-l_1l_2-l_1l_3-l_2l_3$ provided the action is free. These manifolds were introduced by Eschenburg [E], who observed that a certain

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subfamily of those spaces have the nice geometric property admitting a natural riemannian metric of positive sectional curvature. Another family of manifolds of type r is provided by orbit spaces of free S^1 actions $S^1 \times S^5 \times S^3 \to S^5 \times S^3$ on the product of spheres, where we think of S^5 and S^3 as unit spheres in $\mathbb{C}^3$ resp. $\mathbb{C}^2$. The action is defined by

$$(z, (u_1, u_2, u_3), (v_1, v_2)) \mapsto ((z^{k_1} u_1, z^{k_2} u_2, z^{k_3} u_3), (z^{l_1} v_1, z^{l_2} v_2)).$$

The coefficients $k := (k_1, k_2, k_3)$ and $l := (l_1, l_2)$ are assumed to give rise to a free action such that the orbit spaces $M_{k,l}$ are smooth manifolds of type r, which turns out to be equal to $l_1 \cdot l_2$.

The starting-point of the homotopy classification is the observation that to each manifold M of type r we can associate a canonical CW-complex structure which has the form $S^2 \vee S^3 \cup e^4 \cup e^5 \cup e^7$ with attaching maps $\beta_4,\ \beta_5,\ \beta_7$. It can be shown that the 5-skeleton is determined by the algebraic isomorphism type of the Serre spectral sequence of the obvious fibre sequence $G \to M \to BS^1$, and this is completely described by a pair of invariants $\{|r|, s(M)\}$ where $s(M) \in (\mathbb{Z}_r)^*/(\pm 1)$. The number $s(M)$ turns out to be equivalent to the linking form modulo change of orientation.

The most difficult part is the detection of the top cell. If we restrict attention to non-spin manifolds M of type r, with r divisible by 24, it is shown that p_1 mod 24 completely detects the attaching map β_7. In this case we obtain a complete set of invariants $\{|r|, s(M), p_1(M)(24)\}$ (see Theorem 3.4). In general the detection of the top cell is a difficult problem. For the family of spaces $N_{k,l}$ with $k_1 + k_2 + k_3 = l_1 + l_2 + l_3 \equiv 0(3)$ it can be completely solved by considering the obstruction of a certain homotopy lifting problem which gives rise to an unstable invariant $q(N_{k,l})$ that is able to detect the top cell. The calculation of $q(N_{k,l})$ is carried out in Proposition 4.2. The classification Theorem 4.3 can be stated as follows.

Theorem 0.2. *Let $N_{k,l}$, $N_{\tilde{k},\tilde{l}}$ be of type r and assume $\sum k_i = \sum \tilde{k}_i = 0$. The spaces $N_{k,l}$ and $N_{\tilde{k},\tilde{l}}$ are homotopy equivalent if and only if $s(N_{k,l}) = s(N_{\tilde{k},\tilde{l}}) \in (\mathbb{Z}_r)^*/(\pm 1)$ and $q(N_{k,l}) = q(N_{\tilde{k},\tilde{l}}) \in \mathbb{Z}_2$.*

The classification of the family of spaces $M_{k,l}$ of odd type is carried out in terms of the set of invariants $\{s(M_{k,l}), w_2(M_{k,l}), p_1(M_{k,l})(3)\}$ in Theorem 5.1.

Using the fact that the first Pontrjagin class of manifolds is a homeomorphism invariant, we found infinitely many examples for homotopy equivalent but not homeomorphic spaces. Such examples occur even for the subfamily of the $M_{k,l}$ which are homogeneous, i.e. $k_1 = k_2 = k_3$ and $l_1 = l_2$. Note that each homogeneous $M_{k,l}$ admits a canonical riemannian metric with $SU(3) \times SU(2) \times U(1)$ symmetry. This is the reason why they were introduced by Witten [Wi], who was looking for a generalized higher dimensional Kaluza-Klein theory. In dimension < 7 the homotopy type determines the diffeomorphism type in the class of 1-connected compact homogeneous spaces, so these examples are of minimal dimension.

1. General properties

1.1. Let M be a 1-connected, closed 7-dimensional smooth manifold and $r > 0$ an integer. We say M is of type r, iff the cohomology ring of M is of the following form:

$$H^0 = \mathbb{Z}, \quad H^1 = 0, \quad H^2 = \mathbb{Z} \text{ with generator } u,$$
$$H^3 = 0, \quad H^4 = \mathbb{Z}_r \text{ with generator } u^2.$$

We get an invariant for M by the following observation. Consider the S^1-bundle $S^1 \to G \xrightarrow{p} M$ with first Chern class $u \in H^2(M)$. Using the Gysin sequence one shows that $H^*(G)$ is an exterior algebra $E(y_3, y_5)$ in two generators y_3, y_5 of degree 3 and 5. The sequence

$$G \xrightarrow{p} M \xrightarrow{u} BS^1 = K(\mathbb{Z}, 2)$$

is obviously a fibre sequence, so we can study the associated Serre spectral sequence. The differentials in the 4-term $E^4 = E^3 = E^2 = H^*(BS^1) \otimes H^*(G) = \mathbb{Z}[u] \otimes E(y_3, y_5)$ are given by the transgression homomorphism $d^4(1 \otimes y_3) = \pm ru^2 \otimes 1$, so E^5 is the tensor product $E^5 = E^6 = \mathbb{Z}[u]/ru^2 \otimes E(y_5)$. Since u^3 must be killed, there is another differential in the E^6-term,

$$d^6(1 \otimes y_5) = s(M)u^3 \otimes 1,$$

where $s(M) \in (\mathbb{Z}/r)^*$ (:= units in $\mathbb{Z}_r$). The isomorphism type of the Serre spectral sequence is determined by r and $s(M)$, but in order to take care of the choice of the generators we have to factor out the subgroup ± 1 of $(\mathbb{Z}_r)^*$.

Proposition 1.1. *The number $s(M) \in (\mathbb{Z}_r)^*/(\pm 1)$ depends only on the homotopy type of M.*

1.2. The above fibre sequence yields immediately that $p_* : \pi_i(G) \to \pi_i(M)$ is an isomorphism for $i > 2$ and $\pi_2(M) \cong \mathbb{Z}$. Since $H^*(G) \cong H^*(S^3 \times S^5)$ it is easy to check that G is equivalent to a complex

$$S^3 \cup_\alpha e^5 \cup e^8,$$

with attaching map $\alpha \in \pi_4(S^3) \cong \mathbb{Z}_2$. If we assume that $\alpha \neq 0$, then $\pi_4(G) = 0$; consequently the 5-skeleton of G is equal to the 5-skeleton of $SU(3)$. Using the exact sequence of the pair $(G, S^3 \cup e^5)$ and the homotopy groups of $SU(3)$, one observes that the attaching map of the top cell is unique, which proves that G is equivalent to $SU(3)$. Summarizing, we see that M of type r implies that $\pi_4(M) = 0$ or $\pi_4(M) = \mathbb{Z}_2$.

Definition 1.2. If $\pi_4(M) = 0$, we say M is of type A_r; then we have $\pi_2(M) \cong \mathbb{Z}$ and $\pi_i(M) \cong \pi_i(SU(3))$ for $i > 2$. If $\pi_4(M) \cong \mathbb{Z}_2$, we say M is of type B_r; in this case we have $\pi_2(M) \cong \mathbb{Z}$ and $\pi_i(M) \cong \pi_i(S^3 \times S^5)$ for $2 < i < 8$.

1.3. A certain series of manifolds of type r was introduced by Eschenburg [E]. Let $k := (k_1, k_2, k_3)$, $l := (l_1, l_2, l_3) \in \mathbb{Z}^3$ with $k_1 + k_2 + k_3 = l_1 + l_2 + l_3$. Then we obtain an S^1-action on $SU(3)$ by

$$z * A := diag(z^{k_1}, z^{k_2}, z^{k_3})\ A\ diag(z^{-l_1}, z^{-l_2}, z^{-l_3}).$$

If this action happens to be free, which is equivalent to the condition

$$gcd(k_1 - l_{\sigma(1)}, k_2 - l_{\sigma(2)}, k_3 - l_{\sigma(3)}) = 1 \quad \text{for all permutations } \sigma \in S_3,$$

then the orbit space is a smooth manifold, which we denote by $N_{k,l}$. For one-sided actions ($k = (0,0,0)$ or $l = (0,0,0)$) the homogeneous orbit spaces are known as Wallach spaces. The homotopy classification of Wallach spaces was the main motivation for considering manifolds of type r. By using the Serre spectral sequence of the canonical fibration $SU(3) \to N_{k,l} \to BS^1$, one calculates that $N_{k,l}$ is of type $A_{r(k,l)}$ with $r(k,l) := \sigma_2(k_1,k_2,k_3) - \sigma_2(l_1,l_2,l_3)$, where σ_i are the elementary symmetric functions; furthermore

$$s(N_{k,l}) =: s(k,l) = \sigma_3(k_1,k_2,k_3) - \sigma_3(l_1,l_2,l_3) \in (\mathbb{Z}_{r(k,l)})^*/\pm 1.$$

Following Singhof [Si], an easy calculation yields the characteristic classes

$$p_1(N_{k,l}) = (2\sigma_1(k_1,k_2,k_3)^2 - 6\sigma_2(k_1,k_2,k_3))u^2, \quad w_2(N_{k,l}) = 0.$$

Another family of manifolds of type r is provided as orbit spaces of S^1-actions on $S^5 \times S^3$:

$$S^1 \times S^5 \times S^3 \to S^5 \times S^3 \subset \mathbb{C}^3 \times \mathbb{C}^2,$$

$$(z,(u_1,u_2,u_3);(v_1,v_2)) \mapsto (z^{k_1}u_1, z^{k_2}u_2, z^{k_3}u_3);(z^{l_1}v_1, z^{l_2}v_2).$$

If $gcd(k_i,l_j) = 1 \ \forall\ i,j$ and $l_1l_2 \neq 0$, then the action is free and the orbit spaces, which we denote by $M_{k,l}$ for $k := (k_1,k_2,k_3),\ l := (l_1,l_2)$, are of type $B_{r(k,l)}$ with $r(k,l) = l_1l_2$. Similar calculations as above show that

$$s(k,l) = k_1k_2k_3 \in (\mathbb{Z}_{r(k,l)})^*/\pm 1, \quad p_1(M_{k,l}) = (k_1^2 + k_2^2 + k_3^2 - l_1^2 - l_2^2)u^2,$$

$$w_2(M_{k,l}) = (k_1 + k_2 + k_3 + l_1 + l_2)u,$$

$$w_4(M_{k,l}) = (\sigma_2(k_1,k_2,k_3) + \sigma_1(k_1,k_2,k_3)\sigma_1(l_1,l_2))u^2.$$

In case $k_1 = k_2 = k_3$, $l_1 = l_2$ the $M_{k,l}$ are homogeneous spaces, which were also studied by Witten [Wi].

Proposition 1.3. *Let M be of type r. Then M is homotopy equivalent to a CW-complex $S^2 \vee S^3 \cup e^4 \cup e^5 \cup e^7$ with attaching maps β_4, β_5, β_7 for the cells e^4, e^5, e^7. We have $\beta_4 = \nu_2 - r\cdot\iota_3 \in \pi_3(S^2 \vee S^3)$, $\beta_5 = \epsilon\psi + \lambda\cdot[\iota_2,\iota_3] \in \pi_4(S^2 \vee S^3 \cup e^4) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_r$, where ν_2 is the Hopf map, ψ is the image of the generator of $\pi_4(S^3)$ under the inclusion map $S^3 \hookrightarrow S^2 \vee S^3 \cup e^4$, $\epsilon \in \mathbb{Z}_2$ and $\lambda \in (\mathbb{Z}_r)^*$. In case M is of type A_r, β_7 is a generator of a subgroup of $\pi_6(S^2 \vee S^3 \cup e^4 \cup e^5) \cong \mathbb{Z} \oplus \mathbb{Z}_6$ of index 6. In case M is of type B_r, β_7 is a generator of a subgroup of $\pi_6(S^2 \vee S^3 \cup e^4 \cup e^5) \cong \mathbb{Z} \oplus \mathbb{Z}_{12} \oplus \mathbb{Z}_2$ of index* 24.

Proof. Choose generators β_2, β_3 of $\pi_2(M) \cong \mathbb{Z}$ and $\pi_3(M) \cong \mathbb{Z}$. Their wedge product $h_3 := \beta_2 \vee \beta_3 : S^2 \vee S^3 \to M$ is a 3-equivalence. An easy argument involving relative exact sequences and Hurewicz homomorphisms shows that the kernel of $(h_3)_* : \pi_3(S^2 \vee S^3) \to \pi_3(M)$ is generated by $\nu_2 - r\cdot\iota_3$. Therefore we get a 4-equivalence by attaching a 4-cell with attaching map $\beta_4 := \nu_2 - r\iota_3$. It is straightforward to check that we need a 5-cell and a 7-cell to obtain a homotopy equivalence $h : S^2 \vee S^3 \cup e^4 \cup e^5 \cup e^7 \to M$. The corresponding subcomplexes of M will be denoted by M^4, M^5, M^7. Since $\pi_4(M^4) \otimes \mathbb{Q} = \pi_4(S^2) \otimes \mathbb{Q} = 0$ and $\pi_4(M^4, S^2 \vee S^3) \cong \mathbb{Z}$, it follows that $\pi_4(M^4)$ is finite and generated by the image of $[\iota_2,\iota_3]$ and the composition $\psi : S^4 \overset{\nu_3=\Sigma\nu_2}{\longrightarrow} S^3 \hookrightarrow S^2 \vee S^3$ under the inclusion homomorphism $j_* : \pi_4(S^2 \vee S^3) \to \pi_4(M^4)$ (the element $\nu_3 \circ \nu_2 \in \pi_4(S^2)$ is redundant, because of the relation $\nu_2 = r\cdot\iota_3$). By Toda [To], p.65, the Whitehead product $[\iota_2,\nu_2]$ vanishes; therefore $[\iota_2,\iota_3] \in \pi_4(M^4)$ is an r-torsion element and $\beta_4 = \epsilon\psi + \lambda[\iota_2,\iota_3]$ with $\lambda \in \mathbb{Z}_r$. To check that $\pi_4(M^4) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_r$ it remains to show that $\pi_4(M^4)$ has order $2r$. Applying the Hurewicz theorem and the exact sequences of the pair (M^7, M^5), we can reduce this to calculating the homomorphism p_* :

$H_5(G) \to H_5(M)$ (see 1.1), which is multiplication by $\pm r$. Finally we have to calculate $\pi_6(M^5)$; since $p_* : \pi_6(G) \to \pi_6(M)$ is an isomorphism one can use the cellular approximation theorem to see that the short exact sequence

$$0 \to \pi_7(M, M^5) \to \pi_6(M^5) \to \pi_6(M) \to 0$$

splits, which yields the result by applying 1.2. □

Remark 1.4. If M is of type A_r, the exact sequence of the pair (M^5, M^4) shows that $\pi_4(M^4)$ is cyclic; hence $r \equiv 1(2)$ and $\epsilon = 1$ by 1.3.

2. Detecting the 5-cell

In this section we intend to introduce invariants which determine the homotopy type of manifolds of type r up to dimension 5. More precisely, we study the 5-skeleton of the CW-complex that is canonically associated to these special manifolds.

2.1. For the definition of the invariant $s(M)$ we made use of a certain differential in the Serre spectral sequence of the fibre sequence

$$G \xrightarrow{p} M \xrightarrow{u} BS^1,$$

where u induces the homomorphism

$$u^* : H^*(BS^1) \to H^*(M),$$

which maps the universal Chern class c_1 to a generator of $H^2(M)$. This differential is known as the transgression homomorphism (see [Sp], p.518). It can be defined as follows. Consider the homomorphisms

$$H^q(G) \xrightarrow{\delta} H^{q+1}(M, G) \xleftarrow{\tilde{u}^*} H^{q+1}(BS^1) \tag{2.1}$$

for $q > 0$, where δ denotes the connecting homomorphism of the pair (M, G) and M is here identified with the pullback under u of the universal disc bundle over BS^1, which is a manifold with boundary G. The transgression is the homomorphism from a subgroup of $H^q(G)$ to a quotient group of $H^{q+1}(BS^1)$:

$$\tau : \delta^{-1}(Im(\tilde{u}^*)) \to H^{q+1}(BS^1)/ker(\tilde{u}^*), \tag{2.2}$$

defined by $\tau(y) = (\tilde{u}^*)^{-1}\delta(y)$. In our case $q = 5$ and

$$\delta^{-1}(Im(\tilde{u}^*)) = H^5(G).$$

On the other hand we can consider the canonical S^1 bundle

$$S^1 \to G \xrightarrow{p} M,$$

which is classified by the map $u : M \to BS^1$. By integration over the fibres we obtain homomorphisms

$$p_! : H^q(G) \to H^{q-1}(M).$$

For $q = 3, 5$ these are related to τ as follows. Let

$$\Phi : H^*(G) \to H^{*+2}(M, G)$$

be the Thom-homomorphism of the bundle (G, p, M). The homomorphism τ is equal to the composition

$$H^q(G) \xrightarrow{p_!} H^{q-1}(M) \xrightarrow{\Phi} H^{q+1}(M, G) \xleftarrow{\tilde{u}^*} H^{q+1}(BS^1)/ker\tilde{u}^*. \tag{2.3}$$

These homomorphisms can be used to describe the precise relation between the invariant $s(M)$ and the linking form. The linking form is a bilinear form

$$L : H^4(M) \times H^4(M) \to \mathbb{Q}/\mathbb{Z},$$

which can be defined as follows (see [Ba]). Fix an orientation class $\mu \in H_7(M)$. Let x, $y \in H^4(M)$ and $\xi \in H_3(M)$ be the Poincaré dual homology class of y. Moreover let

$$\beta : H^3(M;\mathbb{Q}/\mathbb{Z}) \to H^4(M;\mathbb{Z})$$

be the Bockstein homomorphism which is associated to the exact sequence

$$0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0.$$

Define

$$L(x, y) := \langle \beta^{-1}(x), \xi \rangle \ \in \ \mathbb{Q}/\mathbb{Z}.$$

Since $H^4(M)$ is a cyclic group of order r which is generated by the cohomology class u^2, where u is a generator of H^2, it is clear that the linking form is determined by the number $L(u^2, u^2)$.

The relation between the invariant $s(M)$ and the linking form is given by the following lemma. Note that $L(u^2, u^2)$ is only defined modulo sign, since M is not assumed to be oriented. It is obvious how the definition of the invariant $s(M)$ can be extended to the oriented category so that it becomes strictly equivalent to the linking form.

Lemma 2.1.

$$L(u^2, u^2) = \pm\frac{s^{-1}(M)}{r} \ \in \ \mathbb{Q}/\mathbb{Z},$$

where $s^{-1}(M)$ is some integer such that $s^{-1}(M) \cdot s(M) \equiv \pm 1$ mod r.

Proof. From the fibre bundle $S^1 \to G \xrightarrow{p} M$ we obtain the following commutative diagram of cohomology groups

$$\begin{array}{ccc} H^5(G) & \xrightarrow{p_!} & H^4(M) \\ \downarrow\cong & & \downarrow\cong \\ H_3(G) & \xrightarrow{p_*} & H_3(M), \end{array}$$

where the vertical arrows are the Poincaré duality isomorphisms. Let $x_3 \in H^3(G) \cong \mathbb{Z}$, $x_5 \in H^5(G) \cong \mathbb{Z}$ be generators and $x_3^* \in H_3(G)$ be the dual homology class, e.g. $\langle x_3, x_3^* \rangle = 1$. From (2.3) we obtain $p_!(x_5) = \pm s(M)u^2$. Consequently

$$L(u^2, u^2) = \pm s^{-1}(M)\langle \beta^{-1}(u^2), p_*(x_3^*)\rangle.$$

Recall that, by Proposition 1.3, M is homotopy equivalent to a CW-complex with 4-skeleton

$$M^4 := S^2 \vee S^3 \cup e^4,$$

where e^4 is attached to the 3-skeleton by the map $\beta_4 = \nu_2 - r\iota_3$. In the associated cellular chain complex

$$C_4 \to C_3 \to C_2 \to 0$$

each group is free of rank 1 and there are canonical generators $e^4 \in C_4$, $s^3 \in C_3$, $s^2 \in C_2$ corresponding to the cells of M^4. The boundary homomorphisms of the complex are given by $e^4 \mapsto -rs^3$, $s^3 \mapsto 0$. Let

$$\zeta : C_3 \to \mathbb{Q}/\mathbb{Z}$$

be the homomorphism defined by $\zeta(s^3) := 1/r$. ζ defines a cohomology class $\bar{\zeta} \in H^3(M;\mathbb{Q}/\mathbb{Z})$ such that $\beta(\bar{\zeta}) = -u^2$. Since the cycle s^3 lies in the homology class of $p_*(x_3^*)$, we have

$$\langle \beta^{-1}(u^2), p_*(x_3^*)\rangle = -\langle \zeta, s^3\rangle = -\frac{1}{r}. \qquad \square \tag{2.4}$$

2.2. Consider a manifold M of type r. By Proposition 1.3 we know that M is homotopy equivalent to a CW-complex with the 5-skeleton

$$S^2 \vee S^3 \cup_{\beta_4} e^4 \cup_{\beta_5} e^5,$$

where $\beta_4 := \nu_2 - r\iota_3$ and $\beta_5 := \epsilon\psi + \lambda[\iota_2, \iota_3]$ for some $\epsilon \in \mathbb{Z}_2$, $\lambda \in (\mathbb{Z}_r)^*$. If r is odd we have

$$\epsilon = \begin{cases} 1 & \text{if } M \text{ is of type } A_r, \\ 0 & \text{if } M \text{ is of type } B_r. \end{cases}$$

Consider the case that r is even. Stably we have $\beta_5 = \epsilon \cdot \psi$, because Whitehead products vanish under suspension, and it is obvious that ϵ is detected by $Sq^2 : H^3(M;\mathbb{Z}_2) \to H^5(M;\mathbb{Z}_2)$. Next we will show that the number λ is up to a sign determined by the invariant $s(M)$ or equivalently by the linking form.

Lemma 2.2.

$$L(u^2, u^2) = \pm\rho(\lambda),$$

where $\rho : \mathbb{Z}_r \to \mathbb{Q}/\mathbb{Z}$ is the homomorphism defined by $\rho(n \text{ mod } r) := \frac{n}{r}$.

Proof. We identify M with the CW-complex

$$S^2 \vee S^3 \cup e^4 \cup e^5 \cup e^7$$

with β_4, β_5 as above. Consider the inclusion $h : S^2 \vee S^3 \to M$ of the 3-skeleton and choose $\epsilon_1 \in \mathbb{Z}_2$ in such a way that $\eta := \epsilon_1\psi + [\iota_2, \iota_3]$ vanishes in $\pi_4(M)$. Then there is an extension

$$f : E := S^2 \vee S^3 \cup_\eta e^5 \longrightarrow M$$

of the map h to the Poincaré complex E. Let $\mu_E \in H_5(E)$ be an orientation class. Then $f_*(\mu_E) = l \cdot (u \cap \mu)$ for some $l \in \mathbb{Z}$. The exact sequences of the map $f : (E, S^2 \vee S^3) \to (M, M^4)$ immediately yield $f_*(\eta) = l \cdot (\epsilon\psi + \lambda[\iota_2, \iota_3]) = \epsilon_1\psi + [\iota_2, \iota_3]$; hence $l \equiv \lambda^{-1} \in (\mathbb{Z}_r)^*$. Consider the adjoint homomorphism

$$f_! : H^i(E) \to H^{2+i}(M).$$

The adjunction formula states that

$$f_!(f^*(a) \cdot b) = a \cdot f_!(b)$$

for cohomology classes $a \in H^*(M)$, $b \in H^*(E)$. Let $s^3 \in H_3(M)$ be the homology class of the 3-sphere $S^3 \hookrightarrow S^2 \vee S^3 \cup e^4 \cup e^5 \cup e^7$. Applying the adjunction formula yields

$$s^3 = \pm f_!(f^*(u) \cdot 1) \cap \mu = \pm u \cdot f_!(1) \cap \mu.$$

By definition we have $f_!(1) = l \cdot u$, and consequently

$$s^3 = \pm l \cdot u^2 \cap \mu = \pm\lambda^{-1} u^2 \cap \mu$$

and

$$L(u^2, u^2) = \langle \beta^{-1}(u^2), u^2 \cap \mu\rangle = \pm\langle \beta^{-1}(u^2), \lambda \cdot s^3\rangle = \pm\rho(\lambda).$$

The last equation has been verified in (2.4). □

Consider the complement $M - P$ of a point $P \in M$. It is homotopy equivalent to the 5-skeleton of the CW-complex which we are studying. From Lemma 2.1 and Lemma 2.2 5-cell, $\beta_5 = \epsilon\psi + \lambda[\iota_2, \iota_3]$, is detected up to a sign by the Steenrod square $Sq^2 : H^3 \to H^5$ and the invariant $s(M) \in (\mathbb{Z}_r)^*/\pm 1$. There is an obvious self-equivalence of the 4-skeleton which sends the generator $s^2 \in H_2$ to $-s^2$ and sends β_5 to $-\beta_5$. From this it becomes clear that the invariants $s(M)$ and $Sq^2 : H^3 \to H^5$ completely determine the homotopy types of the complements $M - P$. Hence we can state the following proposition.

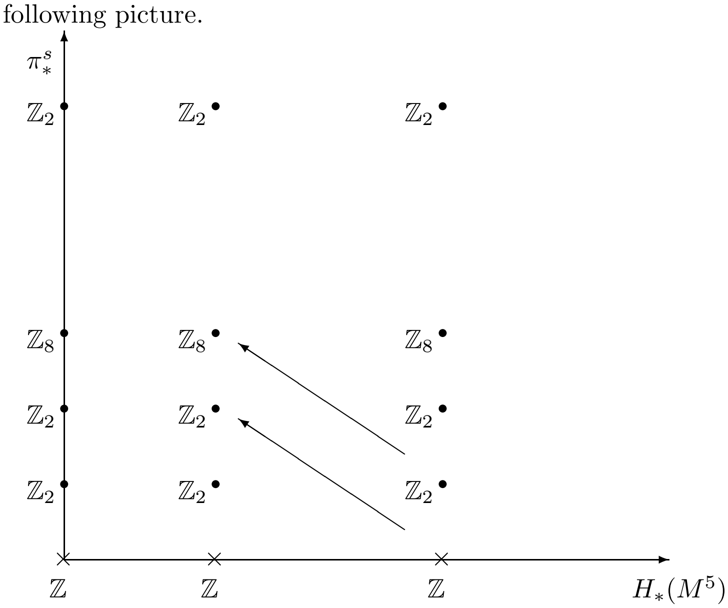
Proposition 2.3. *Let M_1, M_2 be manifolds of the same type A_r or B_r. Suppose further that $Sq^2 : H^3 \to H^5$ and $s(M_1)$, $s(M_2) \in (\mathbb{Z}_r)^*/(\pm 1)$ agree for M_1, M_2. Then $M_1 - P_1$ is homotopy equivalent to $M_2 - P_2$, where $M_i - P_i$ denotes the complement of a point in M_i.*

3. Detecting the 7-cell

3.1. We first study the situation stably in the case when r is not divisible by 2. Consider the Atiyah-Hirzebruch spectral sequence

$$H_p(M^5; \pi_q^s) \Rightarrow \pi_{p+q}^s(M^5)$$

converging to the stable homotopy groups of M^5. Localizing at the prime 2, we get the following picture.



The differentials $d_3 : H_5(M^5; \pi_i^s) \to H_2(M^5; \pi_{i+2}^s)$ depend only on the stable attaching map $\beta_5^s = \epsilon\psi$, which is equal to $\epsilon \cdot \eta^2$ (we identify $\eta^2 \in \pi_4(S^2)$ with its image in M). Therefore we have

$$\tilde{\pi}_6^s(M^5)\otimes\mathbb{Z}_2 = 0 \quad \Leftrightarrow \quad \text{if } M \text{ is of type } A_r,$$
$$\tilde{\pi}_6^s(M^5)\otimes\mathbb{Z}_2 = \mathbb{Z}_2 \quad \Leftrightarrow \quad \text{if } M \text{ is of type } B_r.$$

In the AH spectral sequence converging to $\pi_*^s(M)$, there is another differential $d_2 : H_7(M;\pi_0^s) \to H_5(M;\pi_1^s)$ coming from the stable attaching map β_7. We can identify this differential with the dual of the Steenrod operation in the following sense. Let M^ν be the Thom spectrum of the stable normal bundle of M. It is the Spanier-Whitehead dual spectrum of the CW spectrum M_+ obtained from M by adjoining a base point. The composition of the Thom isomorphism ϕ and the Poincaré duality isomorphism D provides an isomorphism

$$H_k(M) \xrightarrow{D} H^{7-k}(M) \xrightarrow{\phi} \tilde{H}^{7-k}(M^\nu). \tag{3.1}$$

The differential $d_2 : H_7(M;\pi_0^s) \to H_5(M;\pi_1^s)$ corresponds under this isomorphism to the differential

$$d^2 : H^0(M^\nu;\pi^0) \to H^2(M^\nu;\pi^{-1})$$

in the dual AH spectral sequence

$$H^*(M^\nu;\pi^*) \Rightarrow \pi^*(M^\nu)$$

converging to the stable cohomotopy groups of M^ν. Following [AH] this differential is a cohomology operation given by the Steenrod square $Sq^2 : H^0(M^\nu;\mathbb{Z}) \to H^2(M^\nu;\mathbb{Z}_2)$, which acts on the Thom class U_ν by $Sq^2(U_\nu) = w_2U_\nu$. Therefore d_2 vanishes if and only if $w_2(M)$ vanishes, e.g. M is spin. The situation is similar for the 3-primary AH spectral sequence. If r is divisible by 3 there is possibly a differential $d_4 : H_7(M;\pi_0^s) \to H_3(M;\pi_3^s)$, which clearly is dual to the Steenrod operation $P^1 : H^3(M;\mathbb{Z}_3) \to H^7(M;\mathbb{Z}_3)$. By the Wu formula P^1 is just multiplication by the mod 3 reduction of the first Pontrjagin class $p_1(M)$. This completes the discussion of the stable homotopy type when r is not divisible by 2. Summarizing, we have:

Proposition 3.1. *Let $r \equiv 1(2)$. Two manifolds M, M' of type r are stably homotopy equivalent if and only if their cohomology rings are isomorphic as modules over the Steenrod algebras $(H\mathbb{Z}_2)^*(H\mathbb{Z}_2)$, $(H\mathbb{Z}_3)^*(H\mathbb{Z}_3)$ and $\pi_4^s(M)$ is isomorphic to $\pi_4^s(M')$.*

Proposition 3.2. *Each Poincaré complex $X := S^2 \vee S^3 \cup_{\beta_4} e^4 \cup_{\beta_5} e^5 \cup_{\beta_7} e^7$ with attaching maps $\beta_4 := \nu - r\iota_3$, $\beta_5 := \epsilon\psi + \lambda[\iota_2,\iota_3]$, $\epsilon \in \mathbb{Z}_2$, $\lambda \in (\mathbb{Z}_r)^*$, $r \equiv 1(2)$ and β_7 a generator of a subgroup of index* 6 *resp.* 24*, if $\epsilon = 0$ resp.* 1*, is smoothable.*

Proof. The stable vector bundles over X are classified by w_2 and p_1. This can be seen as follows. First observe that $\tilde{KO}(X) = [X, BO]$ has no contribution from the 5- and 7-cells. Consider the 4-skeleton $X^4 := S^2 \vee S^3 \cup e^4$ of X. The cofibration

$$S^3 \xrightarrow{g} X^4 \longrightarrow \mathbb{P}^2\mathbb{C},$$

where the map g represents a generator of $\pi_3(X^4)$, gives rise to an exact sequence

$$0 \to \tilde{KO}^{-1}(S^3) \to \tilde{KO}^0(\mathbb{P}^2) \to \tilde{KO}^0(X^4) \to 0.$$

Since the stable vector bundles over $\mathbb{P}^2$ are classified by w_2 and p_1 the same is true for X^4. Choose a stable vector bundle ξ with $w_2(\xi) = w_2(X)$ and $p_1(\xi)$ mod 3 $= -v_1(X)$, where $v_1(X) :=$ is the first Wu class of X. Using the Thom isomorphism, one easily verifies that the Thom spectrum $M(\xi)$ of ξ has the property that $Sq^2 : H^5(M(\xi)) \to H^7(M(\xi))$ and $P^1 : H^3(M(\xi);\mathbb{Z}_3) \to H^7(M(\xi);\mathbb{Z}_3)$ both vanish (the Thom class lies by convention in H^0). Since r is odd this implies that the stable top cell of $M(\xi)$ splits off, so the top homology class is spherical and the claim follows by classical surgery theory. □

Remark 3.3. For arbitrary r the stable homotopy type should have a classification in terms of the $(H\mathbb{Z}_2)^*(H\mathbb{Z}_2)$ module structure of the cohomology ring and the first Pontrjagin class reduced modulo 24. The problem is that p_1 modulo 24 has to be defined in terms of stable homotopy theory. This would also lead to a generalization of Proposition 3.2. We will discuss this point in a later paper.

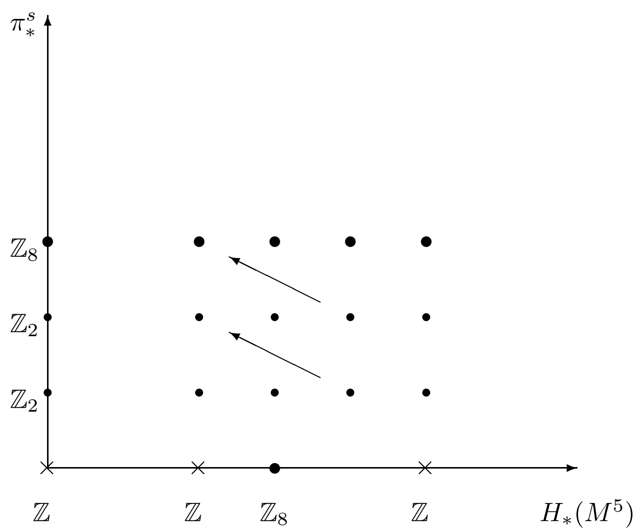
There is one case where the nonstable homotopy type is determined by the known invariants.

Theorem 3.4. *Let M, M' be two manifolds of type r such that $w_2 \neq 0$ for both of them, and let $r \equiv 0(24)$. Then M and M' are homotopy equivalent if and only if $s(M) = s(M') \in (\mathbb{Z}_r)^*/(\pm 1)$ and $p_1(M) = p_1(M')$ mod 24.*

Proof. M, M' are of type B_r by Remark 1.4. We show that the attaching maps β_7, β_7' of M, M' are homotopic. Consider the Atiyah-Hirzebruch spectral sequence

$$H_p(M^5, \pi_q^s) \Rightarrow \pi_{p+q}^s(M^5)$$

converging to the stable homotopy groups of M^5. Localizing at the prime 2, we have the following picture:



where the small dots represent the group $\mathbb{Z}_2$ and the big dots represent the group $\mathbb{Z}_8$. The only differentials in this picture correspond under the isomorphism (3.1) to the cohomology differentials in the AH spectral sequence

$$H^p(M^\nu; \pi^q) \Rightarrow \pi^{p+q}(M^\nu)$$

converging to the stable cohomotopy groups of the Thom spectrum M^ν of the stable normal bundle of M. These differentials can be identified by [AH] with the Steenrod square Sq^2. The action of Sq^2 on $H^*(M^\nu)$ is given by the formula

$Sq^2(U_\nu) = w_2 U_\nu$, where U_ν denotes the Thom class of the stable normal bundle ν. As the referee pointed out to me, in the case when r is even, $w_2 \neq 0$ is equivalent to the vanishing of $Sq^2 : H^3 \to H^5$. For let $u \in H^2$ and $x \in H^3$ be the generators of the cohomology ring with $\mathbb{Z}_2$ coefficients. By the Wu formula

$$w_2 ux = Sq^2(ux) = u^2 x + uSq^2(x);$$

hence $w_2(M) \neq 0$ is equivalent to $Sq^2(x) = 0$. This implies $Sq^2(w_2 U_\nu) = 0$ and $Sq^2(xU_\nu) = xw_2U_\nu \neq 0$. Therefore the differentials

$$d_2 : H_4(M^5; \pi_i^s) \to H_2(M^5; \pi_{i+1}^s)$$

are the only non-trivial differentials in the picture. Since the stable 5-cell splits off, we have $\pi_6^s(M^5) \cong \mathbb{Z}_8 \oplus \mathbb{Z}_2$. One easily calculates that $Tors(\pi_6(M^5)) \to \pi_6^s(M^5)$ is a monomorphism with image $\mathbb{Z}_4 \oplus \mathbb{Z}_2$. Identify the 5-skeletons of M and M', and let ν, ν' be the stable normal bundles of M, M'. Since $w_2(M) = w_2(M')$ and $p_1(M) = p_1(M')$ mod 24, the spherical fibrations that are associated to ν, ν' are equivalent over the 5-skeletons. The top cells of the Thom spectra M^ν and $M^{\nu'}$ split off, and there is a splitting preserving homotopy equivalence $h : M^\nu = (M^5)^\nu \vee S^7 \to (M^5)^{\nu'} \vee S^7 = M^{\nu'}$. Clearly, the stably dual homotopy equivalence $h^* : M'_+ \to M_+$ restricted to the 5-skeleton sends β_7' to $\pm\beta_7$. □

3.2. So far we have studied only the stable attaching map for the top cell. We now consider the Whitehead product

$$\pi_2(M^5) \times \pi_5(M^5) \to \pi_6(M^5),$$

which is clearly an unstable invariant. By 1.2 we have

$$\pi_5(M^5) \cong \begin{cases} \mathbb{Z} & \text{type } A_r, \\ \mathbb{Z} \oplus \mathbb{Z}_2 & \text{type } B_r. \end{cases}$$

Let $\lambda_5 \in \pi_5(M^5)$ be a generator if M is of type A_r, or a generator of a subgroup of index 2 if M is of type B_r.

Lemma 3.5. *Let M be of type r with $r \equiv 1(2)$. Then*

$$[\iota_2, \lambda_5] = \pm(\mu \cdot \beta_7 + \alpha),$$

where α, μ do not depend on the choice of λ_5, α is a torsion element, and $\mu = 2r$ if M is of type A_r, otherwise $\mu = r$.

Proof. The determination of μ is a straightforward calculation. We need to show that α does not depend on the choice of λ_5. Let $0 \neq \delta \in \pi_5(S^3) = Tors(\pi_5(M^5)) \cong \mathbb{Z}_2$, $\delta := E(\nu \circ E\nu)$, where ν is the Hopf map and E is the suspension. Then

$$[\iota_2, \delta] = [\iota_2 \circ E(\iota_1), \iota_3 \circ E(\nu \circ E(\nu))] = [\iota_2, \iota_3] \circ E^2(\nu \circ E(\nu)).$$

By Proposition 1.3 and the fact $r \equiv 1(2)$ we obtain $[\iota_2, \iota_3] = 0$. □

4. Classification results

In this section we classify a subseries of the generalized Wallach spaces consisting of those spaces $N_{k,l}$ with $k_1 + k_2 + k_3 = l_1 + l_2 + l_3 \equiv 0(3)$. Observe that the transformation $(k_i, l_j) \mapsto (k_i', l_j')$, where $k_i' := k_i - (k_1 + k_2 + k_3)/3$ and $l_j' := l_j - (l_1 + l_2 + l_3)/3$, does not change the S^1-action. Consequently we have $N_{k,l} = N_{k',l'}$ and $\sum k_i' = \sum l_j' = 0$.

4.1. Assume from now on that $\sum k_i = \sum l_j = 0$ and that the corresponding action is free. Let $\varphi_{k,l} : S^1 \to SU(3) \times SU(3)$ be the homomorphism

$$z \mapsto \{diag(z^{k_1}, z^{k_2}, z^{k_3}), diag(z^{l_1}, z^{l_2}, z^{l_3})\}.$$

There is an obvious diffeomorphism of the double coset space

$$\varphi_{k,l}(S^1)\backslash SU(3) \times SU(3)/\Delta(SU(3))$$

with $N_{k,l}$. The quotient map

$$p : \varphi_{k,l}(S^1)\backslash SU(3) \times SU(3) \to N_{k,l}$$

defines a fibre bundle with fibre $SU(3)$. Let $\iota_2' \in \pi_5(\varphi_{k,l}(S^1)\backslash SU(3) \times SU(3)) \cong \mathbb{Z}$ and $\lambda_5' \in \pi_5(SU(3))$ be generators and let

$$\Psi : (\varphi_{k,l}(S^1)\backslash SU(3) \times SU(3)) \times SU(3) \longrightarrow \varphi_{k,l}(S^1)\backslash SU(3) \times SU(3)$$

be the map $\Psi([A, B], C]) := [AC, B]$. The composition

$$p \circ \Psi \circ (\iota_2' \times \lambda_5') : S^2 \times S^5 \to N_{k,l}$$

is a map which represents on each factor a generator of $\pi_2(N_{k,l})$ resp. $\pi_5(N_{k,l})$. Together with Lemma 3.5 this shows that

$$[\iota_2, \lambda_5] = \pm 2r\beta_7 \ \in \ \pi_6(N_{k,l}).$$

In the case that r is not divisible by 3, this determines β_7 up to 2-torsion, but this is also true for $r \equiv 0(3)$ since $p_1(N_{k,l}) \equiv 0(3)$ detects the attaching map 3-locally (3.1). We now come to the hardest point in classifying the series $N_{k,l}$, that is, the calculation the 2-primary part of β_7. The idea is to consider for a certain space of type A_1, say N_0, the obstruction for the existence of a map $h : N_0 \to N_{k,l}$, such that $h^* : H^2(N_{k,l}) \to H^2(N_0)$ is an isomorphism. The most convenient choice for N_0 is $N_0 := N_{(0,0,0);(1,0,-1)}$. It has the property that it admits self-maps of N_0 of arbitrary odd degree in $H^5(N_0)$, each of which induces the identity on $H^2(N_0)$. These maps can be constructed by using the canonical action by left translations $SU(3) \times N_0 \to N_0$ to multiply maps $N_0 \to SU(3)$ (of arbitrary even degree in H^5) with the identity of N_0.

Proposition 4.1. *Two spaces $N_{k,l}$, $N_{\tilde{k},\tilde{l}}$ are homotopy equivalent if and only if the following conditions hold.*

1. $s(k, l) := s(N_{k,l}) = s(\tilde{k}, \tilde{l}) \ \in \ (\mathbb{Z}_r)^*/\pm 1$.
2. *For both spaces there exist maps $h_1 : N_0 \to N_{k,l}$, $h_2 : N_0 \to N_{\tilde{k},\tilde{l}}$ inducing an isomorphism in H^2, or*

2'. *For both spaces there exist no maps as in 2.*

Proof. Assume that the conditions 1 and 2 hold. By Proposition 1.3 we can assume that the 5-skeletons of both spaces are identical, say to a complex $N^5 = S^2 \vee S^3 \cup e^4 \cup e^5$. Let $\beta_{7,0}$, β_7, $\tilde{\beta}_7$ be the attaching maps of the top cells of N_0, $N_{k,l}$, $N_{\tilde{k},\tilde{l}}$ respectively. We have

$$\left.\begin{array}{rcl} (h_1)_*(\beta_{7,0}) & = & s_1\beta_7 \\ (h_2)_*(\beta_{7,0}) & = & s_2\tilde{\beta}_7 \end{array}\right\} \in \ \pi_6(N^5),$$

for some s_1, $s_2 \in \mathbb{Z}$. Composing h_1 and h_2, if necessary, with appropriate self-maps of N_0, we can assume $s_1 = s_2 =: s$. The integer s must be odd; this is because s is also the degree of the homomorphism $H_5(N_0) \to H_5(N_{k,l})$, which is the same as the degree of $\pi_5(N_0^5, N_0^4) \to \pi_5(N^5, N^4)$. Since $\pi_4(N_0^4) \to \pi_4(N^4)$ is clearly a monomorphism, we can conclude by Proposition 1.3 and by the exactness of the relative homotopy sequences that s must be an odd number. Now it suffices to show that the restricted maps $h_1 \mid N_0^5$, $h_2 \mid N_0^5$ are homotopic, because in this case $s\beta_7 = s\tilde{\beta}_7$ and consequently $\beta_7 = \tilde{\beta}_7$. This last assertion is an easy exercise, since N_0^5 can be interpreded as the 2-cell complex $S^2 \cup e^5$. The proof is similar in the case when the conditions 1 and 2' hold. □

Following Singhof [Si], the space $N_{k,l} = \varphi_{k,l}(S^1)\backslash SU(3) \times SU(3)/\Delta SU(3)$ is a pullback up to homotopy:

$$\begin{array}{ccc} N_{k,l} & \longrightarrow & BSU(3) \\ \downarrow & & \downarrow \Delta \\ BS^1 & \xrightarrow{B\varphi_{k,l}} & BSU(3) \times BSU(3), \end{array}$$

where the vertical maps are the diagonal map Δ and a map representing a generator of $H^2(N_{k,l})$. The existence of a map $N_0 \to N_{k,l}$ inducing an isomorphism in H^2 is equivalent to the existence of a lift of the map $u : N_0 \to BS^1$, representing a generator in $H^2(N_0)$. Write $\varphi_{k,l} = (\varphi_k, \varphi_l) : S^1 \to SU(3) \times SU(3)$.

There is a lift of the map $u : N_0 \to BS^1$ if and only if the two compositions $B\varphi_k \circ u$, $B\varphi_l \circ u : N_0 \to BS^1 \to BSU(3)$ are homotopic.

Proposition 4.2. *The compositions $B\varphi_k \circ u$, $B\varphi_l \circ u$ are homotopic if and only if*

$$q(k) = q(l) \ \in \ \mathbb{Z}_2,$$

where $q(k) := \frac{1}{2}(-\sigma_2(k_1, k_2, k_3)^2 - \sigma_2(k_1, k_2, k_3))$ mod 2.

As a consequence of Propositions 4.1 and 4.2 we get the classification theorem. Let $q(k, l) := q(k) + q(l) \in \mathbb{Z}_2$.

Theorem 4.3. *Let $N_{k,l}$, $N_{\tilde{k},\tilde{l}}$ be of type r and let $\sum k_i = \sum \tilde{k}_i = 0$. The spaces $N_{k,l}$ and $N_{\tilde{k},\tilde{l}}$ are homotopy equivalent if and only if*

$$s(k, l) = s(\tilde{k}, \tilde{l}) \ \in \ (\mathbb{Z}_r)^*/\pm 1 \quad \text{and} \quad q(k, l) = q(\tilde{k}, \tilde{l}) \ \in \ \mathbb{Z}_2.$$

Corollary 4.4. *The Wallach spaces $N_{(0,0,0);(l_1,l_2,l_3)}$, $N_{(0,0,0);(\tilde{l}_1,\tilde{l}_2,\tilde{l}_3)}$ are homotopy equivalent if and only if*
$l_1^2 + l_1 l_2 + l_2^2 = \tilde{l}_1^2 + \tilde{l}_1\tilde{l}_2 + \tilde{l}_2^2 =: r$ *and* $l_1 l_2(l_1 + l_2) = \pm\tilde{l}_1\tilde{l}_2(\tilde{l}_1 + \tilde{l}_2) \in \mathbb{Z}_r$.

Example 4.5. Let $k := (n + 1, n - 1, -2n)$, $l := (n, n, -2n)$. The corresponding S^1-action is free and the spaces $N_{k,l}$ are all of type A_1. We have $s(k, l) = 1 \bmod 1$ and

$$q(k, l) = \begin{cases} 0 & \text{if } n \text{ is even,} \\ 1 & \text{if } n \text{ is odd.} \end{cases}$$

As a consequence of Lemma 3.5 we conclude that there are two distinct homotopy types of type A_1 with $[\iota_2, \lambda_5] = 0$. These two types are represented by the spaces $N_0 = N_{(0,0,0);(1,0,-1)}$ and $N_{(1,1,-2);(2,0,-2)}$.

Example 4.6. The following spaces are of type A_7:

k	l	$s(k,l)$	$q(k,l)$	$p_1(k,l)$
$(18,-8,-10)$	$(17,-13,-4)$	3	0	1
$(0,0,0)$	$(3,-2,-1)$	6	1	0
$(-4,-2,6)$	$(-5,1,4)$	5	0	0
$(21,-7,-14)$	$(16,-20,4)$	6	1	0
$(44,10,-54)$	$(47,5,-52)$	3	0	2
$(62,-36,-26)$	$(61,-41,-20)$	3	0	4

There are three pairs (k,l) with $s(k,l)=3(7)$ and $q(k,l)=0(2)$. The corresponding spaces are therefore homotopy equivalent. Since $p_1(k,l)$ are pairwise distinct, the spaces are not homeomorphic.

The proof of the following two lemmas is left to the reader.

Lemma 4.7. *There is a 7-equivalence $S^4\cup e^6\to BSU(3)$, where $S^4\cup e^6$ is the 2-cell complex whose 6-cell is attached by the non-trivial element of $\pi_5(S^4)$. Furthermore the inclusion homomorphism $\pi_7(S^4)\to\pi_7(S^4\cup e^6)$ is an epimorphism and $\pi_7(S^4\cup e^6)\cong\mathbb{Z}\oplus\mathbb{Z}_6$.*

Lemma 4.8. *Let $k:N_0\to N_0/N_0^5=S^7$ be the collapsing map. Then $k^*:[S^7,BSU(3)]\to[N_0,BSU(3)]$ is a* 1-1 *correspondence.*

Proof of Proposition 4.2. N_0 is the total space of a fibre bundle

$$U(2)/S^1\to N_0\xrightarrow{\pi}SU(3)/U(2),$$

where $U(2)\hookrightarrow SU(3)$ is embedded by

$$A\mapsto\begin{pmatrix}A & \\ & det(A^{-1})\end{pmatrix}.$$

The quotient spaces are standard: $SU(3)/U(2)=\mathbb{P}^2\mathbb{C}$, $U(2)/S^1=S^3$. The composition $N_0\to\mathbb{P}^2\mathbb{C}\to\mathbb{P}^\infty\mathbb{C}=BS^1$, which is induced by π, is homotopic to the map u representing a generator of H^2. Since the classifying space $BSU(3)$ is 3-connected and $H^4(BSU(3))$ is generated by the second Chern class c_2 of the universal bundle, two maps $f_1,\ f_2:\mathbb{P}^2\mathbb{C}\to BSU(3)$ are homotopic if and only if $f_1^*(c_2)=f_2^*(c_2)$ in $H^4(\mathbb{P}^2\mathbb{C})$. Let $r_1:=-\sigma_2(k_1,k_2,k_3)$, $r_2:=-\sigma_2(l_1,l_2,l_3)$ and let $L_r:\mathbb{P}^2\mathbb{C}\to BSU(3)$ be a map with $L_r^*(c_2)=ru^2$, where $u\in\ H^2(\mathbb{P}^2\mathbb{C})$ is a generator. It remains to show that $L_{r_1}\circ\pi$ and $L_{r_2}\circ\pi$ are homotopic if and only if

$$\frac{1}{2}(-r_1^2+r_1)\equiv\frac{1}{2}(-r_2^2+r_2)\ \mathrm{mod}\ 2.$$

Let E, F be the pullbacks of the universal $SU(3)$-bundle by the 7-connected map $j:S^4\cup e^6\to BSU(3)$ and $L_{-1}:\mathbb{P}^2\mathbb{C}\to BSU(3)$. We obtain the following diagram:

$$\begin{array}{ccccccc}
 & & SU(3) & & SU(3) & & SU(3)\\
 & & \downarrow & & \downarrow & & \downarrow\\
 & & E & \xrightarrow{\hat{L}_{-1}} & F & \longrightarrow & *\\
 & & \downarrow & & \downarrow & & \downarrow\\
N_0 & \xrightarrow{\pi} & \mathbb{P}^2\mathbb{C} & \xrightarrow{L_{-1}} & S^4\cup e^6 & \xrightarrow{j} & BSU(3).
\end{array}$$

The composition $L_{-1}\circ\pi$ is clearly homotopic to a constant, since N_0 is the homotopy fibre of the map $B\varphi_{(1,0,-1)} : BS^1 \to BSU(3)$. Consequently π has a lift $\hat{\pi} : N_0 \to E$. Using the commutative diagram of fibre sequences

$$\begin{array}{ccc} S^3 & \longrightarrow & SU(3) \\ \downarrow & & \downarrow \\ N_0 & \xrightarrow{\hat{L}_{-1}\circ\hat{\pi}} & F \\ \downarrow & & \downarrow \\ \mathbb{P}^2\mathbb{C} & \xrightarrow{L_{-1}\circ\pi} & S^4\cup e^6, \end{array}$$

one easily calculates that F is 6-connected, $H^7(F)\cong\mathbb{Z}$ and $\hat{L}_{-1}\circ\hat{\pi} : H^7(F) \to H^7(N_0)$ is an isomorphism. Let F_0 be the pullback of the bundle $SU(3)\to F\to S^4\cup e^6$ by the inclusion $S^4 \hookrightarrow S^4\cup e^6$. Since $\pi_4(BSU(2))\cong\pi_4(BSU(3))$ we can reduce the structure group of $SU(3)\to F_0\to S^4$ to $SU(2)$; hence we obtain another diagram

$$\begin{array}{ccc} S^3 & & SU(3) \\ \downarrow & & \downarrow \\ S^7 & \xrightarrow{\kappa} & F \\ \downarrow & & \downarrow \\ S^4 & \longrightarrow & S^4\cup e^6, \end{array}$$

where the left fibre map is the Hopf map. Clearly κ is an 8-equivalence. For dimensional reasons the map $\hat{L}_{-1}\circ\hat{\pi}$ factors over κ and, consequently, the compositions

$$N_0 \xrightarrow{\pi} \mathbb{P}^2\mathbb{C} \xrightarrow{L_{-1}} S^4\cup e^6,$$

$$N_0 \xrightarrow{k} S^7 \xrightarrow{\nu} S^4\cup e^6$$

are homotopic, where ν denotes the Hopf map composed with the inclusion. Every degree r map $S^4\to S^4$ can be extended to a self-map of $S^4\cup e^6$, which we simply denote by r. Therefore the three maps

$$L_{-r}\circ\pi,\ r\circ L_{-1}\circ\pi,\ r\circ\nu\circ k : N_0 \to S^4\cup e^6$$

are homotopic. Applying Lemma 4.8, we simply have to prove that the maps $j\circ r_1\circ\nu$, $j\circ r_2\circ\nu$ are homotopic if and only if $\frac{1}{2}(-r_1^2+r_1) = \frac{1}{2}(-r_2^2+r_2) \bmod 2$. First calculate the endomorphism

$$r_* : \pi_7(S^4) \to \pi_7(S^4),$$

induced by the degree r map. Let a_4 be a generator of $Tors(\pi_7(S^4))\cong\mathbb{Z}_{12}$—this is clearly the suspension of an element $a_3\in\pi_6(S^3)$. Let $r_*(\nu_4) = r^2\nu_4 + \lambda(r)\cdot a_4$. By Toda [To] we have $[\iota_4,\iota_4] = 2\nu_4 - a_4$, so

$$r_*[\iota_4,\iota_4] = r^2[\iota_4,\iota_4] = r^2(2\nu_4 - a_4).$$

On the other hand

$$\begin{aligned} r_*[\iota_4,\iota_4] &= r_*(2\nu_4 - a_4) \\ &= 2r^2\nu_4 + (2\lambda(r) - r)a_4; \end{aligned}$$

consequently

$$2\lambda(r) - r \equiv -r^2 \bmod 12.$$

By Lemma 4.7 the kernel of $\pi_7(S^4) \to \pi_7(BSU(3))$ is generated by ν and $6a_4$, so

$$j \circ r \circ \nu = \lambda(r) \cdot j \circ a_4,$$

where $j \circ a_4$ generates $\pi_7(BSU(3)) \cong \mathbb{Z}_6$ and $\lambda(r) = -\frac{1}{2}(-r^2 + r) \in \mathbb{Z}_6$. For $r = \sigma_2(k_1, k_2, k_3)$ an easy argument shows that $\lambda(r) \equiv 0(3)$ for all possible k_i's. □

5. Further results

The spaces $M_{k,l}$ (see 1.3) admit a map $g_{k,l} : M_{k,l} \to S^2$ unless $l_1 l_2 \neq 0$. This further information allows us to classify the series $M_{k,l}$ at least for $r(k,l) \equiv 1(2)$.

5.1. Consider the S^1-action on S^3:

$$S^1 \times S^3 \to S^3, \quad (u, v) \mapsto (z^{l_1} u, z^{l_2} v).$$

Although the action is not free (unless $l_1,\ l_2 \in\ \pm 1$) the quotient space S^3/S^1 is homeomorphic to S^2. Furthermore the projection $pr : S^5 \times S^3 \to S^3$ induces a map $g_{k,l} : M_{k,l} \to S^2$. The Hopf invariant of the quotient map can easily be calculated to be $\pm l_1' l_2'$, where $l_i' := l_i/q$ and $q := gcd(l_1, l_2)$. An easy consequence is that the homomorphism $(g_{k,l})_* : H_2(M_{k,l}) \to H_2(S^2)$ is multiplication by $\pm t(k,l)$, where $t(k,l) := l_1 l_2 / q$.

Theorem 5.1. *Let $r \equiv 1(2)$ and let $M_{k,l}$, $M_{\tilde{k},\tilde{l}}$ be of type r. Then $M_{k,l}$ is homotopy equivalent to $M_{\tilde{k},\tilde{l}}$ if and only if*

$$\begin{aligned} s(M_{k,l}) &= s(M_{\tilde{k},\tilde{l}}) \ \in\ (\mathbb{Z}_r)^*/\pm 1, \\ w_2(M_{k,l}) &= w_2(M_{\tilde{k},\tilde{l}}), \\ p_1(M_{k,l}) &= p_1(M_{\tilde{k},\tilde{l}}) \ mod\ 3. \end{aligned}$$

Proof. no By Lemma 3.5 we have

$$\pm r \cdot \beta_7 - [\iota_2, \lambda_5] \ \in\ \text{torsion} \ \cap\ ker(g_{k,l})_*.$$

Since r is odd, $\pm r\beta_7 - [\iota_2, \lambda_5]$ is detected by $w_2(M_{k,l})$, except in the case when r is divisible by 3. Consequently β_7 is detected by $w_2(M_{k,l})$ and in addition $p_1(M_{k,l})$ mod 3 (see 3.2). □

Corollary 5.2. *(The Homogeneous Case-Witten Spaces) Let $k_1 = k_2 = k_3$, $l_1 = l_2$, $\tilde{k}_1 = \tilde{k}_2 = \tilde{k}_3$, $\tilde{l}_1 = \tilde{l}_2$, $gcd(k_i, l_j) = 1$. The spaces $M_{k,l}$ and $M_{\tilde{k},\tilde{l}}$ are homotopy equivalent if and only if $l_1^2 = \tilde{l}_1^2$ and $k_1^3 = \pm\tilde{k}_1^3$ mod $2l_1^2$.*

Corollary 5.3. *Same assumptions as in 5.2 and $l_1,\ \tilde{l}_1 \neq 0(3)$. Then $M_{k,l}$ and $M_{\tilde{k},\tilde{l}}$ are homeomorphic if and only if they are homotopy equivalent and their first Pontrjagin classes are equal.*

Proof. By Kreck and Stolz [KS1] $M_{k,l}$, $M_{\tilde{k},\tilde{l}}$ are homeomorphic if and only if $l_1^2 = \tilde{l}_1^2$ and $k_1 = \pm\tilde{k}_1$ mod $2l_1^2$. By 1.3 we have $p_1(M_{k,l}) = 3k_1^2 u^2$. Since $l_1 = \pm\tilde{l}_1$ is not divisible by 3, $3k_1^2 = \tilde{k}_1^2 \in \mathbb{Z}_{l_1^2}$ implies $k_1^2 = \tilde{k}_1^2$. □

Example 5.4. Corollary 5.3 is not true if $l_1 = \pm\tilde{l}_1$ is divisible by 3. Let k_i, $\tilde{k}_i$, l_i, $\tilde{l}_i$ be as in 5.3 and let $l_1 = \tilde{l}_1 = 3$, k_1, $\tilde{k}_1 \in\ 1, 5, 7$ mod 18; then $M_{k,l}$ and $M_{\tilde{k},\tilde{l}}$ are homotopy equivalent. If in addition $k_1 \neq \tilde{k}_1$ mod 18, then $M_{k,l}$ and $M_{\tilde{k},\tilde{l}}$ are not homeomorphic although their first Pontrjagin classes are equal.

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